Hierarchy of random deterministic chaotic maps with an invariant measure

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Abstract

Hierarchy of one and many-parameter families of random trigonometric chaotic maps and one-parameter random elliptic chaotic maps of **cn** type with an invariant measure have been introduced. Using the invariant measure (Sinai-Ruelle-Bowen measure), the Kolmogrov-Sinai entropy of the random chaotic maps have been calculated analytically, where the numerical simulations support the results .

Keywords: Chaos, random chaotic dynamical systems, Invariant measure, Kolmogorov-Sinai entropy, Lyapunve exponent .

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1 Introduction

The problem of the transition to chaos in deterministic systems has been the subject of much interest, and, for low-dimensional dynamics, it has been found that this transition most often occurs via a small number of often observed routes (e.g., period doubling and intermittency). Usually the analytic calculation of invariant measure of dynamical systems is a nontrivial task, hence there are limited number of maps with invariant measure like, Ulam-Von Neumann map [1], chebyshev maps [2], Katsura-Fukuda map [3], piecewise parabolic map [4], Tent map[5], Elliptic map [6] and finally hierarchy of one and many-parameter families of random trigonometric chaotic and one-parameter random elliptic chaotic maps of **cn** type and their coupling [8, 9, 7, 10].
Here in this work we give a new hierarchy of random chaotic maps with an invariant measure, where using the invariant measure we discuss analytically the transition to chaos in these random dynamical systems. Random maps have attracted the attention of physicists in the realm of theoretical biology, disordered systems, and cellular automata for its possible application to studies of DNA replication, cell differentiation, and evolution theory [11, 12]. Additionally, random maps are of interest as models of convection by temporarily irregular fluid flows [13].
In this paper we consider random maps where, on each iteration, the map function $\Phi(x, \alpha)$ is chosen randomly from a hierarchy of chaotic maps with an invariant measure which are introduced in our previous papers [8, 9, 10].
There are two noticeable advantages of random chaotic maps that are presented through this article. First they are measureable dynamical system so they can be studied analytically. Second, they have the property of being either chaotic or having stable period one fixed point.
The paper is organized as follows. Section 2 is devoted to introduction of the random map models. Then, in the section 3, we introduce Sinai-Ruelle-Bowen (SRB) measure of random chaotic maps. The calculation of Kolmogorov-Sinai (KS) entropy of random chaotic maps via their SRB-measure is presented in section 4. Section 5 is devoted to the Lyapunov characteristic exponent. The paper is ended with a brief conclusion in section 6.

2 Hierarchy of one and many-parameter random chaotic maps of trigonometric and elliptic types

The random chaotic map can be obtained via random choice from an ensemble of chaotic maps according to some probability distribution. Therefore for a given ensemble of chaotic maps $\Phi_i, \ i = 1, 2, ..$ with probability $p_i \geq 0$ with $\sum p_i = 1$, the corresponding random map can be defined as

$$\Phi(x, p) = \Phi_i(x) \qquad \text{with probability } p_i \tag{2.1}$$

Here in this paper we try to constructs hierarchy of random chaotic maps with an invariant measure by choosing the ensemble of one and many parameters of chaotic maps of trigonometric and elliptic types of references [8, 9, 10] as follows.

2.A One-parameter random trigonometric maps

The families of one-parameter trigonometric chaotic maps of the interval $[0,1]$ are defined as the ratio of hypergeometric polynomials through the following equation [8]:

$$\Phi_N(x,\alpha) = \frac{\alpha^2\left(1+(-1)^N{}_2F_1(-N,N,\frac{1}{2},x)\right)}{(\alpha^2+1)+(\alpha^2-1)(-1)^N{}_2F_1(-N,N,\frac{1}{2},x)}. \tag{2.2}$$

where N is an integer greater than one, obviously these map the unit interval $[0,1]$ into itself. Using the hierarchy of families of one-parameter chaotic maps (2.1), we can generate a new hierarchy of random maps with an invariant measure, denote by $\Phi(p_i, N_i, \alpha, x)$ which can be written as:

$$\Phi\left(p_i, N_i, \alpha, x\right) = \Phi_{N_i}\left(a_{N_i}(\alpha), x\right) \qquad \text{with probability } p_i, \tag{2.3}$$

where $\sum_{i=1}^m p_i = 1$ and the parameter $a_N(\alpha)$ are defined as

$$a_N(\alpha) = \frac{\sum_{k=0}^{[\frac{(N-1)}{2}]} C_{2k+1}^N \left(\frac{\alpha}{1-\alpha}\right)^{-k}}{\sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N \left(\frac{\alpha}{1-\alpha}\right)^{-k}}, \tag{2.4}$$

with the symbol $[\,]$ as greatest integer part, and $C_m^n = \frac{n!}{m!(n-m)!}$.

2.B Many-parameter random trigonometric maps

Even thought one can define many-parameters random trigonometric chaotic maps with an invariant measure, but for simplicity we restrict ourselves here in this paper to two-parameters ones [9]. Random two-parameters trigonometric chaotic maps are defined as:

$$\Phi(P_{ij}, \alpha_i, \alpha_j, N_i, N_j, x) = \Phi_{N_i N_j}\left(\alpha_i, \alpha_j, x\right) \qquad \text{with probability } p_{ij}, \tag{2.5}$$

where $\sum_{i,j} P_{ij} = 1$.

$$\Phi_{N_i N_j}(\alpha_i, \alpha_j, x) = \Phi_{N_i}(\Phi_{N_j}(x, \alpha_j), \alpha_i) \tag{2.6}$$

with

$$\alpha_i^{-1} = \alpha_j \times \frac{A_{N_j}(\alpha)}{B_{N_j}(\alpha)} \times \frac{A_{N_i}\left(\frac{1}{\eta_{N_j}^{\alpha_j}(\alpha)}\right)}{B_{N_i}\left(\frac{1}{\eta_{N_j}^{\alpha_j}(\alpha)}\right)} \tag{2.7}$$

and

$$\eta_{N_j}^{\alpha_j}(\alpha) = \alpha_j \times \left(\frac{\alpha}{1-\alpha}\right) \times \left(\frac{A_{N_j}(\alpha)}{B_{N_j}(\alpha)}\right)^2, \tag{2.8}$$

where the polynomials $A(x)$, $B(x)$ are defined as:

$$A(x) = \sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N \left(\frac{x}{1-x}\right)^k,$$

$$B(x) = \sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N \left(\frac{x}{1-x}\right)^k. \tag{2.9}$$

2.C One-parameter random elliptic maps

The families of one-parameter elliptic chaotic maps of **cn** [14] at the interval $[0,1]$ are defined as the ratio of Jacobian elliptic functions of **cn** types in the following form [10]:

$$\Phi_N^e(x,\alpha) = \frac{\alpha^2\left(cn\left(N cn^{-1}(\sqrt{x})\right)\right)^2}{1+(\alpha^2-1)\left(cn(N cn^{-1}(\sqrt{x}))\right)^2}. \tag{2.10}$$

Obviously these map the unit interval $[0,1]$ into itself. Now, with the hierarchy of families of one-parameter elliptic chaotic maps (2.9), we can generate a new hierarchy of one-parameter random elliptic maps with an invariant measure, denote by $\Phi^e(p_i,\alpha_i,x)$ which can be written as:

$$\Phi^e(p_i,N_i,\alpha,x) = \Phi^e_{N_i}(a_{N_i}(\alpha),x) \qquad \text{with probability } p_i, \tag{2.11}$$

where $\sum_{i=1}^m p_i = 1$ and $a_{N_i}(\alpha)$ is the same as given in (2.12).

3 Invariant measure of random chaotic maps

Characterizing invariant measure for a given nonlinear dynamical systems is a fundamental problem which connects dynamical theory to statistics and mechanics. A well-known example is Ulam and von Neumann map which has an ergodic measure $\mu = \frac{1}{\sqrt{x(1-x)}}$ [1]. Let us recall that for a deterministic map $\Phi(x)$, the invariant probability measure $\mu(x)$ is the eigenfunction of the Perron-Frobenius(PF) operator **L** related to maximum eigenvalue 1 [15, 16]

$$L\mu(x) = \mu(x), \tag{3.1}$$

where the operator L is defined as:

$$Lf(x) = \int \delta(y-\Phi(x))f(y)dy = \sum_{z=\Phi^{-1}(x)} \frac{f(z)}{\mid \Phi'(z)\mid}. \tag{3.2}$$

In the case of random map, the average probability density can be found by the straightforward generalization of (3.1) as follows

$$\bar{L}\mu_{av}(x) = \mu_{av}(x), \tag{3.3}$$

where

$$\bar{L} = \sum_{i=1}^m p_i L_i, \tag{3.4}$$

where L_i is the Perron-Frobenius operator associated with map $\Phi_i(x)$.
It should be mentioned that for trigonometric chaotic maps [8], their composition [9] and their coupling [7] the eigenstate of PF operator **L** corresponding to largest eigenvalues has already been obtained in our previous papers. Now,we choose the hierarchy of trigonometric chaotic maps $\Phi(N_i,\alpha,x)$, as the ensemble of chaotic maps. Then $\Phi(p_i,N_i,\alpha,x)$-invariance of average density $\mu_{av}(x,\alpha)$ implies that the average density should satisfies the following formal Perron-Frobenius(PF) integral equation

$$\mu_{av}(p,y,\alpha) = \sum_{i=1}^m p_i \int_0^1 \delta\left(y-\Phi_{N_i}(a_{N_i}(\alpha),x)\right)\mu_i(x,\alpha)dx. \tag{3.5}$$

Obviously above equation is the generalization of Equation (3.2) for random trigonometric chaotic maps. As it is shown in[8], each integral appearing on the right hand side of (3.5) can be written as

$$\mu_i(y, \alpha) = \sum_{x_{ij} \varepsilon \Phi^{-1}_{N_i}(y, a_{N_i}(\alpha))} \mu_i(x_{ij}, \alpha) dx_j. \tag{3.6}$$

Using the prescription of Reference [8] one can show that $\mu_i(x, \alpha)$, the invariant measure associated with trigonometric chaotic maps $\Phi(N_i, \alpha, x)$ has the following form

$$\mu_i(x, \alpha) = \mu(x, \alpha) = \frac{1}{\pi} \frac{\sqrt{\frac{\alpha}{1-\alpha}}}{\sqrt{x(1-x)}\left(\frac{\alpha}{1-\alpha} + (1 - \frac{\alpha}{1-\alpha})x\right)}, \tag{3.7}$$

that is, the invariant measure $\mu_i(x, \alpha)$ given in (3.7) satisfies equation (3.6). Now, multiplying both side of equation (3.6) by p_i and summing over i we get

$$\mu_{av}(p, x, \alpha) = \sum_{i=1}^{m} p_i \mu_i(x, \alpha) = \mu(x, \alpha) = \sum_{i=1}^{m} p_i \times \sum_{x_{ij} \varepsilon \Phi^{-1}(p_i, N_i, \alpha, y)} \mu_i(x, \alpha) dx_{ij} \tag{3.8}$$

Therefore, the density $\mu(x, \alpha)$ given in (3.7) is the average invariant measure for ensemble of trigonometric chaotic maps $\Phi_i(N_i, x)$ and it satisfies PF-equation (3.5), hence $\mu_{av}(x, \alpha)$ is the invariant or SRB measure [16, 17] of random trigonometric chaotic maps maps given in (2.4) define on the interval $[0, 1]$. Also, as the relation (3.7) shows $\mu_{av}(x, \alpha) = \mu_i(x, \alpha)$, hence the average invariant measure for random trigonometric chaotic maps is equal to the invariant measure of each map of ensemble of chaotic maps.
Also one can show that the average density $\mu_{av}(x, \alpha)$ given in (3.7) has the following asymptotic form of delta function as α goes to zero and one, respectively,that is, we have

$$\mu_{av}(x, \alpha)_{\alpha \to 0} = \delta(x), \tag{3.9}$$

$$\mu_{av}(x, \alpha)_{\alpha \to 1} = \delta(x - 1). \tag{3.10}$$

where the first one corresponds to invariant measure associated with fixed point $x = 0$ and the latter one corresponds to the fixed point $x = 1$. It is straight forward to show that the random trigonometric chaotic maps are well defined for $\alpha > 1$, where they have fix point $(x = 1)$[8], therefore, they posses Dirac delta function invariant measure for $\alpha > 1$, too.
Similarly one can show that the average density of two-parameters(many-parameters) random trigonometric chaotic maps is the same as the average invariant measure $\mu_{av}(x, \alpha)$ given in (3.7).
Finally in the case of elliptic random maps, as it is shown in reference [10], for small values of elliptic parameter **k**, elliptic chaotic maps are topologically conjugated with trigonometric chaotic maps. Hence, for small **k** the average invariant measure for one-parameter random elliptic chaotic maps of **cn** type, is also the same as the average invariant measure $\mu_{av}(x, \alpha)$ given in (3.7).

4 Kolmogrov-Sinai entropy of random chaotic maps

KS-entropy or metric entropy [16] measures how chaotic a dynamical system is and it is proportional to the rate at which information about the state of dynamical system is

lost in the course of time or iteration. Therefore, it can also be defined as the average rate of loss of information for a discrete measurable dynamical system $(\Phi(x,p),\mu_{av})$. By introducing a partition $\alpha = A_c(n_1, ...n_\gamma)$ of the interval $[0,1]$ into individual laps A_i, one can define the usual entropy associated with the partition by:

$$H(\mu_{av},\gamma) = -\sum_{i=1}^{n(\gamma)} m(A_c)\ln m(A_c),$$

where $m(A_c) = \int_{n\in A_i}\mu_{av}(x,\alpha)dx$ is the invariant measure of A_i. Defining a n-th refining $\gamma(n)$ of γ:

$$\gamma^n = \bigcup_{k=0}^{n-1}(\Phi(x,p))^{-(k)}(\gamma)$$

then an entropy per unit step of refining is defined by :

$$h(\mu_{av},\Phi(x,p),\gamma) = \lim{}_{n\to\infty}\left(\frac{1}{n}H(\mu_{av},\gamma)\right),$$

now, if the size of individual laps of $\gamma(N)$ tends to zero as n increases, the above entropy reduces to well known as KS-entropy, that is:

$$h(\mu_{av},\Phi(x,p)) = h(\mu_{av},\Phi(x,p),\gamma).$$

KS-entropy is actually a quantitative measure of the rate of information lost with the refining and it can be written as [15]:

$$h(\mu_{av},\Phi(x,p) = \sum_{i=1}^{m} p_i\int \mu_{av}(x)\ln|\frac{d\Phi_i(x)}{dx}|dx, \tag{4.1}$$

which is also a statistical mechanical expression of the Lyapunov Characteristic exponent, that is: mean divergence rate of two nearby orbits. The measurable random dynamical system $(\Phi(p,x),\mu_{av})$ is chaotic for $h(\mu_{av},\Phi(x,p)) > 0$ and predictive for $h(\mu_{av},\Phi(x,p)) = 0$. Using the fact that the invariant measure for these random chaotic maps is equal to invariant measure of each map of ensemble of chaotic maps, one can show that KS-entropy of these random chaotic maps is the average KS-entropy of chaotic maps of ensemble, that is we have:

$$h(\mu_{av},\Phi(x,p)) = \sum_{i=1}^{m} p_i\times h(\mu_i,\Phi_i(x)). \tag{4.2}$$

4.A KS-entropy of one-parameter random trigonometric chaotic maps

With a prescription similar to the prescription of Reference [8], one can calculate KS-entropy of hierarchy of trigonometric chaotic maps $\Phi(a_{N_i}(\alpha),x)$, where we quote only the result below

$$h\left(\mu_i,\Phi(N_i,\alpha,x)\right) = \ln\left(\frac{N_i(\frac{1}{1-\alpha}+2\sqrt{\frac{\alpha}{1-\alpha}})^{N_i-1}}{(\sum_{k=0}^{[\frac{N_i}{2}]}C_{2k}^{N_i}(\frac{\alpha}{1-\alpha})^k)(\sum_{k=0}^{[\frac{N_i-1}{2}]}C_{2k+1}^{N_i}(\frac{\alpha}{1-\alpha})^k)}\right). \tag{4.3}$$

Therefore, substituting for KS-entropy of one-parameter trigonometric chaotic map in Equation (4.1), we get the following expression for KS-entropy of one-parameter random trigonometric chaotic maps $h\left(\mu_{av}, \Phi(p_i, N_i, \alpha, x)\right)$

$$h\left(\mu_{av}, \Phi(p_i, N_i, \alpha, x)\right) = \sum_{i=1}^{m} p_i \ln\left(\frac{N_i(\frac{1}{1-\alpha} + 2\sqrt{\frac{\alpha}{1-\alpha}})^{N_i-1}}{(\sum_{k=0}^{[\frac{N_i}{2}]} C_{2k}^{N_i}(\frac{\alpha}{1-\alpha})^k)(\sum_{k=0}^{[\frac{N_i-1}{2}]} C_{2k+1}^{N_i}(\frac{\alpha}{1-\alpha})^k)}\right). \quad (4.4)$$

Using the asymptotic Dirac delta function form of the average density $\mu_{av}(x, \alpha)$ for limiting values of $\alpha = o$, and 1 given in (3.9) and (3.10), respectively, one can show that, KS-entropy of one-parameter random trigonometric chaotic maps takes the following form:

$$h\left(\mu_{av}, \Phi(p_i, N_i, \alpha, x)\right) = \sum_{i=1}^{m} p_i \ln\mid \frac{d\Phi_{N_i}\left(a_{N_i}(\alpha), x\right)}{dx}|_{x=0}\mid = \sum_{i=1}^{m} p_i \ln\mid \frac{N_i}{a_{N_i}^2(\alpha)}\mid = 0. \quad (4.5)$$

as $\alpha \to 0$, and

$$h\left(\mu_{av}, \Phi(p_i, N_i, \alpha, x)\right) = \sum_{i=1}^{m} p_i \ln\mid \frac{d\Phi_{N_i}(a_{N_i}(\alpha), x)}{dx}|_{x=1}\mid = \sum_{i=1}^{m} p_i \ln\mid N_i a_{N_i}^2(\alpha)\mid. \quad (4.6)$$

as $\alpha \to 1$ and for $\alpha > 1$, respectively.
It is straight forward to see that each summed on the right hand side of (4.4) has the asymptotic form $(1-\alpha)^{\frac{1}{2}}$ as $\alpha \to 1_-$. Thus $h(\mu_{av}, \Phi(p_i, N_i, \alpha, x))$ has the following asymptotic form as $(\alpha \to 1_-)$.

$$\begin{cases} h\left(\mu_{av}, \Phi\left(p_i, N_i, \alpha \to 1_-, x\right)\right) \sim (1-\alpha)^{\frac{1}{2}}, \\ \\ h\left(\mu_{av}, \Phi\left(p_i, N_i, \alpha \to 0_+, x\right)\right) \sim (\alpha)^{\frac{1}{2}}, \end{cases} \quad (4.7)$$

The above asymptotic form indicates that the maps $\Phi(p_i, N_i, \alpha, x)$ belong to the same universality class which are different from the universality class of pitch fork bifurcating maps but their asymptotic behavior is similar to class of intermittent maps[18], even though intermittency can not occur in these maps for any values of parameter $a_N(\alpha)$, since the maps $\Phi(p_i, N_i, \alpha, x)$ and their n-composition $\Phi^{(n)}$ do not have minimum values other than zero and maximum values other than one in the interval $[0, 1]$.

4.B KS-entropy of many-parameter random trigonometric chaotic maps

Similarly, one can calculate KS-entropy of many-parameter random chaotic maps with the prescription reference [9]:

$$h\left(\mu_{av}, \Phi(p_{ij}, N_i, N_j, \alpha_i, \alpha_j, x)\right) =$$

$$\sum_{i,j} p_{ij} \ln\left(\frac{N_i N_j(1+\sqrt{\frac{1-\alpha}{\alpha}})^{2(N_j-1)}(1+\sqrt{\eta_{N_j}^{\alpha_j}(\alpha)})^{2(N_i-1)}}{A_{N_j}(\alpha)B_{N_j}(\alpha)A_{N_i}(\eta_{N_j}^{\alpha_j}(\alpha))B_{N_i}(\eta_{N_j}^{\alpha_j}(\alpha))}\right). \quad (4.8)$$

With respect to the one-parameter random trigonometric chaotic maps, the numerical and theoretical calculations predict different asymptotic behavior for many-parameter

random trigonometric chaotic maps, as example of asymptotic of the composed maps ($\phi_{2,3}(\alpha_1, \alpha_2, x)$ and $\phi_{3,2}(\alpha_1, \acute{\alpha}_2, x)$), the KS-entropy $h(\mu_{av}, \Phi)$ is presented below:

$$h(\mu_{av}, \Phi) = p\times\ln\left(\frac{3\left((1-\alpha)+\sqrt{\alpha(1-\alpha)}\right)^4\left((2\alpha+1)(1-\alpha)+\alpha_2^2(3-2\alpha)\sqrt{\alpha(1-\alpha)}\right)^2}{(1-\alpha)^3(1+2\alpha)(3-2\alpha)((1-\alpha)(1+2\alpha)^2+\alpha_2^2\alpha(3-2\alpha)^2)}\right)+$$

$$(1-p)\times\ln\left(\frac{3((1-\alpha)+\sqrt{\alpha(1-\alpha)})^4((2\alpha+1)(1-\alpha)+\acute{\alpha}_2^2(3-2\alpha)\sqrt{\alpha(1-\alpha)})^2}{(1-\alpha)^3(1+2\alpha)(3-2\alpha)((1-\alpha)(1+2\alpha)^2+\acute{\alpha_2}^2\alpha(3-2\alpha)^2)}\right) \quad (4.9)$$

with the following relation among the parameters

$$\alpha_2^{-1} = \frac{\alpha_1(1-\alpha)(1+2\alpha)^3}{2(3-2\alpha)((1-\alpha)(1+2\alpha)-\alpha_1\alpha(3-2\alpha)^2)}$$

$$\acute{\alpha}_2^{-1} = \frac{\alpha_1(1-\alpha)(1+2\alpha)^3}{2(3-2\alpha)((1-\alpha)(1+2\alpha)-\alpha_1\alpha(3-2\alpha)^2)}$$

which is obtained from the relation (2.8). Now choosing $\alpha_2 = \acute{\alpha}_2$ and $\alpha = \frac{\alpha_2^{\nu}}{1+\alpha_2^{\nu}}, 0 < \nu < 2$, the entropy given by (4.9) reads:

$$h = p\times\ln\left(\frac{3(1+\sqrt{\alpha_2^{\nu}})^4(1+3\alpha_2^{\nu})(1+\alpha_2^{\nu+2}\sqrt{\alpha_2^{\nu}})}{(3+\alpha_2^{\nu})(1+3\alpha_2^{\nu})((3\alpha_2^{\nu}+1)+\alpha_2^{\nu+2}(3+\alpha_2^{\nu})^2)}\right)+$$

$$(1-p)\times\ln\left(\frac{3(1+\sqrt{\alpha_2^{\nu}})^4(1+3\alpha_2^{\nu})(1+\alpha_2^{\nu+2}\sqrt{\alpha_2^{\nu}})}{(3+\alpha_2^{\nu})(1+3\alpha_2^{\nu})((3\alpha_2^{\nu}+1)+\alpha_2^{\nu+2}(3+\alpha_2^{\nu})^2)}\right) \quad (4.10)$$

which has the following asymptotic behavior

$$\begin{cases} h(\mu_{av}, \Phi) \sim \alpha_2^{\frac{\nu}{2}} & \text{as} \quad \alpha_2 \longrightarrow 0(\alpha \longrightarrow 0), \\ h(\mu_{av}, \Phi) \sim (\frac{1}{\alpha_2})^{\frac{\nu}{2}} & \text{as} \quad \alpha_2 \longrightarrow \infty(\alpha \longrightarrow 1). \end{cases}$$

The above asymptotic behaviours indicate that for an arbitrary value of $0 < \nu < 2$ the maps $\Phi(p_{ij}, 2, 3, \alpha_1, \alpha_2, x)$ belong to the universality class which is different from the universality class of one-parameter trigonometric chaotic maps of $\Phi(p_i, N_i, \alpha, x)$ (2.2) or the universality class of pitch fork bifurcating maps.

4.C KS-entropy of one-parameter random elliptic chaotic maps

For random one-parameter elliptic chaotic maps of **cn** type, KS-entropy, for small values of elliptic parameter by considering random elliptic chaotic maps are topologically conjugated with random trigonometric chaotic maps [10] would be equal to KS-entropy of one-parameter random trigonometric chaotic maps, is represented with (4.3).

5 Lyapunov exponent of random chaotic maps

The Lyapunov exponent λ provides the simplest information about chaoticty and can be computed considering the separation of two nearby trajectories evolving in the same realization of the random process, and for random chaotic maps given in $(2.3, 5, 11)$, it can be defined as [19]

$$\lambda(x_0) = lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln \mid \frac{d\Phi(x,p)}{dx} \mid, \tag{5.11}$$

where $x_k = \overbrace{\Phi_N \circ \Phi_N \circ .... \circ \Phi_N^k(x_0)}$. It is obvious that its negative values, indicate that the system is in fix point (attractor)regime, while its positive values indicate that the system is measurable(the Invariant measure given in $(3.9, 12)$) [19]. Also, the lyapunov number is independent of initial point, provided that the motion inside the invariant manifold is ergodic, thus $\lambda(x_0)$ characterizes the invariant manifold of random map as a whole. For values of parameter α, such the map $\Phi(x,p)$ be measurable, Birkhof ergodic [20] theorem implies the equality of KS-entropy and lyapunov number, i.e.,

$$h(\mu_{av}, \Phi(x,p)) = \lambda(x_0)$$

Also comparing KS-entropy of these maps by their Lyapunov exponent confirms this prediction (see figures 1a,1b,2a and 2b). In chaotic region, random maps are ergodic as Birkhof ergodic theorem predicts. In non-chaotic region of the parameter, lyapunov characteristic exponent is negative definite, since in this region, we have only single period fixed points without bifurcation.

6 Conclusion

In this paper we have discussed the dynamical characterization of systems whose evolution is described by random maps. We have studied the application of Perron-Frobenius operator to the analysis of the dynamical behaviour of random dynamical systems in order to derive the invariant measure of the system. Again this interesting property is due to the existence of invariant measure for a region of the parameters space of these maps.

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Figure Captions

Fig.1-a. Shows the variation of KS-entropy of one-parameter random trigonometric chaotic map for ensemble of $(\Phi_2(x, a_2(\alpha))$ and $\Phi_3(x, a_3(\alpha)))$ in term of the parameter α and p

Fig.1-b. Shows the variation of Lyapunov characteristic exponent of one-parameter random trigonometric chaotic map for ensemble of $(\Phi_2(x, a_2(\alpha))$ and $\Phi_3^e(x, a_3(\alpha)))$ in term of the parameter α and p

Fig.2-a. Shows the variation of KS-entropy of one-parameter random elliptic chaotic maps for ensemble of $(\Phi_2^e(x, a_2(\alpha))$ and $\Phi_3^e(x, a_3(\alpha)))$ in term of the parameter α and p

Fig.2-b. Shows the variation of Lyapunov characteristic exponent of one-parameter random elliptic chaotic maps for ensemble of $(\Phi_2^e(x, a_2(\alpha))$ and $\Phi_3^e(x, a_3(\alpha)))$ in term of the parameter α and p d

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